

SPECTRAL RELATIONSHIPS OF MIXED INTEGRAL EQUATION IN THREE DIMENSIONS

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Abstract

Here, the solution of the mixed integral equation of the first kind in one, two, and three dimensions is obtained in the space $L_2[\Omega] \times C[0, T]$, $T < 1$; Ω is the domain of integration with respect to position. Many spectral relationships, when the kernel of the position takes a form of logarithmic kernel, Carleman kernel, elliptic kernel, potential kernel, generalized potential kernel, and Macdonald kernel, are considered as special cases of this work. Also, many important cases are obtained and discussed.

1. Introduction

The mathematical formulation of physical phenomena, population genetics, mechanics, and contact problems in the theory of elasticity, often involves singular integral equation with different kernels. Over the past 40 years, substantial progress has been in developing approximate

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analytical and purely numerical solution to a large class of singular integral equation of the first and second kind.

The monographs [1-4, 7] contain many different spectral relationships for different kinds of integral equations, in one, two, and three dimensions. Also, in [12, 13], using Krein's method, Mkhitarian and Abdou obtained the spectral relationships for the integral operator containing logarithmic kernel and Carleman function, respectively.

In this work, the spectral relationships for the mixed integral equation in one, two, and three dimensionals, will be obtained. The results of the previous works in this domain will be considered as special cases of this work. Many important cases will be discussed, when the kernel of the position takes different forms of Weber-Sonien integral.

2. Mixed Integral Equation

Consider the mixed integral equation

$$\lambda_1 \int_{\Omega} k(x, y) \Phi(y, t) dy + \lambda_2 \int_0^t \int_{\Omega} F(|t - \tau|) k(x, y) \Phi(y, \tau) dy d\tau = f(x, t),$$

$$(x = \bar{x}(x_1, x_2, x_3), \quad y = \bar{y}(y_1, y_2, y_3)), \quad (2.1)$$

under the condition

$$\int_{\Omega} \Phi(x, t) dx = P(t). \quad (2.2)$$

Here, the given function $f(x, t)$ is defined in the space $L_2[\Omega] \times C[0, T]$, $T < 1$; Ω is the domain of integration with respect to position. The pressure $P(t)$, $t \in [0, T]$, $T < 1$, and λ_1 and λ_2 are constants, may be complex, and having many physical meaning. The unknown function $\Phi(x, t)$ will be obtained in the space $L_2[\Omega] \times C[0, T]$, $T < 1$; the known function $k(x, y)$ is the kernel of the position and has a singular term, while $F(|t - \tau|)$ is the kernel of Volterra integral term, measured with respect to the time, and represents the resistance of the layer of the surface against the pressure $P(t)$.

In order to guarantee the existence of unique solution of (2.1), we assume the following conditions:

(i) The kernel of the position, $k(x, y)$, $x = \bar{x}(x_1, x_2, x_3)$, and $y = \bar{y}(y_1, y_2, y_3)$, satisfies in $L_2(\Omega)$, where Ω is the domain of integration with respect to the position, the discontinuity condition, Fredholm condition

$$\left\{ \int_{\Omega} \int_{\Omega} k^2(x, y) dx dy \right\}^{\frac{1}{2}} = A, \quad (A \text{ is a constant}).$$

(ii) The positive continuous function $F(|t - \tau|) \in C([0, T] \times [0, T])$, and satisfies $|F(|t - \tau|)| < B$, B is a constant, for all values $(t, \tau) \in [0, T]$.

(iii) The given function $f(x, t)$ with its first partial derivatives are continuous and belong to the class $L_2(\Omega) \times C[0, T]$, where its norm is defined as

$$\|f\|_{L_2 \times C} = \max_{0 \leq t \leq T} \int_0^t \left\{ f^2(x, \tau) dx \right\}^{\frac{1}{2}} d\tau, \quad x = \bar{x}(x_1, x_2, x_3).$$

(iv) The unknown function $\Phi(x, t)$ satisfies Hölder condition until respect to the time and Lipschitz condition with respect to the position.

The integral equation (2.1) can be adapted in the integral operator form

$$K\Phi = f, \quad K\Phi = \lambda_2 L\Phi + \lambda_1 M\Phi, \quad (2.3)$$

where

$$L\Phi = \int_{\Omega} k(x, y) \Phi(y, t) dy, \quad (2.4)$$

and

$$M\Phi = \int_0^t \int_{\Omega} F(|t - \tau|) k(x, y) \Phi(y, \tau) dy d\tau, \quad (2.5)$$

$$(x = \bar{x}(x_1, x_2, x_3), \quad y = \bar{y}(y_1, y_2, y_3)).$$

The continuity of the integral operator (2.3) can be proved by taking two sets $u_1, u_2 \in [\Omega]$, $u = \bar{u}(x_1, x_2, x_3)$ to have

$$\begin{aligned} |K_1\Phi - K_2\Phi| &\leq |\lambda_1| |L_1\Phi - L_2\Phi| + |\lambda_2| |M_1\Phi - M_2\Phi| \\ &= |\lambda_2| \left| \int_{\Omega} k(u_1, y)\Phi(y, t)dy - \int_{\Omega} k(u_2, y)\Phi(y, t)dy \right| \\ &\quad + |\lambda_1| \left| \int_0^t \int_{\Omega} F(|t - \tau|)k(u_1, y)\Phi(y, \tau)dy d\tau \right. \\ &\quad \left. - \int_0^t \int_{\Omega} F(|t - \tau|)k(u_2, y)\Phi(y, \tau)dy d\tau \right|. \end{aligned}$$

Using Cauchy-Schwartz inequality, and conditions (i) and (iii), we get

$$|K_1\Phi - K_2\Phi| \leq [(|\lambda_2| + B|\lambda_1|T)g(u_1, u_2)] \cdot \|\Phi\|,$$

where $T = \max_{0 \leq t \leq T} t$ and $g(u_1, u_2) = \left(\int_{\Omega} |k(u_1, y) - k(u_2, y)|^2 dy \right)^{\frac{1}{2}}$. Hence, when $u_1 \rightarrow u_2$, we have $g(u_1, u_2) \rightarrow 0$, which leads us to write $K_1\Phi \rightarrow K_2\Phi$. The normality of the integral operator (2.3) can be proved as

$$\begin{aligned} \|K\Phi\|_{L_2 \times c} &\leq |\lambda_2| \|L\Phi\| + |\lambda_1| \|M\Phi\| \\ &= |\lambda_2| \left\| \left\{ \int_{\Omega} k^2(x, y) dx dy \right\}^{\frac{1}{2}} \left\{ \int_{\Omega} \Phi^2(y, \tau) dy \right\}^{\frac{1}{2}} d\tau \right\| \\ &\quad + |\lambda_1| B \left\| \left\{ \int_{\Omega} k^2(x, y) dx dy \right\}^{\frac{1}{2}} \cdot \max \int_0^t \left\{ \int_{\Omega} \Phi^2(y, \tau) dy \right\}^{\frac{1}{2}} d\tau \right\|. \end{aligned}$$

Using Cauchy-Schwartz inequality, and conditions (i) and (iii), we have

$$\|K\Phi\| \leq (|\lambda_2| + B|\lambda_1|)TA \|\Phi\| = \gamma \|\Phi\|,$$

where

$$\gamma = (|\lambda_2| + B|\lambda_1|)TA < 1.$$

3. System of Fredholm Integral Equations

If we divide the interval $[0, T]$, $0 \leq t \leq T < 1$, as $0 \leq t_0 < t_1 < \dots < t_i = T$, when $t = t_\ell$, $\ell = 0, 1, 2, \dots, i$. The integral equation (2.1) takes the form

$$\begin{aligned} \lambda_1 \int_0^{t_1} \int_{\Omega} F(|t_\ell - \tau|) k(x, y) \Phi(y, \tau) dy d\tau \\ + \lambda_2 \int_{\Omega} k(x, y) \Phi(y, t_\ell) dy = f(x, t_\ell), \end{aligned} \quad (3.6)$$

which can be adapted in the form

$$\begin{aligned} \lambda_1 \sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{\Omega} k(x, y) \Phi_j(y) dy + \lambda_2 \int_{\Omega} k(x, y) \Phi_\ell(y) dy + O(h_\ell^{p+1}) = f_\ell(x), \\ (h_\ell \rightarrow 0, \quad p > 0), \end{aligned} \quad (3.7)$$

where $h_\ell = \max_{0 \leq j \leq \ell} h_j$ and $h_j = t_{j+1} - t_j$.

Here, we used the following notations:

$$F(|t_\ell - t_j|) = F_{\ell,j}, \quad \Phi(y, t_\ell) = \Phi_\ell(y), \quad f(x, t_\ell) = f_\ell(x). \quad (3.8)$$

The values u_j and the constant \bar{p} depend on the number of derivatives of $F(|t - \tau|)$ with respect to t , see [8].

Also, the boundary condition (2.2) becomes

$$\int_{\Omega} \phi_\ell(x) dx = P_\ell, \quad \ell = 0, 1, 2, \dots, N. \quad (3.9)$$

The formula (3.2) represents a linear system of Fredholm integral equations of the first kind, where its solution depends on the kind of the kernel $k(x, y)$ and the domain of integration Ω . In the application, we will neglect $O(h_\ell^{p+1})$.

4. Spectral Relationships

In this section, we will obtain spectral relationships for mixed integral equation in one, two, and three dimensions.

4.1. Potential kernel in finite domain

Assume the domain of integration Ω , in (3.7), in the form $\Omega = \{(x, y, z) \in \Omega : \sqrt{x^2 + y^2} \leq a, z = 0\}$ and the kernel of the position takes the potential function form

$$k(x - \xi, y - \eta) = [(x - \xi)^2 + (y - \eta)^2]^{-\frac{1}{2}}. \quad (4.10)$$

Hence, we have the following system:

$$\lambda_1 \sum_{j=0}^{\ell} u_j F_{j,k} \int_{\Omega} \frac{\Phi_j(\xi, \eta) d\xi d\eta}{\sqrt{(x - \xi)^2 + (y - \eta)^2}} + \lambda_2 \int_{\Omega} \frac{\Phi_{\ell}(\xi, \eta) d\xi d\eta}{\sqrt{(x - \xi)^2 + (y - \eta)^2}} = f_{\ell}(x, y). \quad (4.11)$$

Using the polar coordinates, the formula (4.11) takes the form

$$\begin{aligned} \lambda_1 \sum_{j=0}^{\ell} u_j F_{j,k} \int_0^a \int_{-\pi}^{\pi} \frac{\rho \Phi_j(\rho, \psi) d\rho d\psi}{\sqrt{r^2 + \rho^2 - 2r\rho \cos(\theta - \psi)}} \\ + \lambda_2 \int_0^a \int_{-\pi}^{\pi} \frac{\rho \Phi_{\ell}(\rho, \psi) d\rho d\psi}{\sqrt{r^2 + \rho^2 - 2r\rho \cos(\theta - \psi)}} = f_{\ell}(r, \theta). \end{aligned} \quad (4.12)$$

Here, we used the following notations:

$$\begin{aligned} x &= r \cos \theta, & y &= r \sin \theta, & \xi &= \rho \cos \psi, & \eta &= \rho \sin \psi, \\ \psi_j(x, y) &= \psi_j(r \cos \theta, r \sin \theta) = \psi_j(r, \theta). \end{aligned} \quad (4.13)$$

To separate the variables, one assumes

$$\Phi_j(r, \theta) = \Phi_j^{(m)}(r) \begin{cases} \cos m\theta \\ \sin m\theta \end{cases}, \quad f_{\ell}(r, \theta) = f_{\ell}^{(m)}(r) \begin{cases} \cos m\theta \\ \sin m\theta \end{cases}. \quad (4.14)$$

Using (4.14) in (4.12), we have

$$\lambda_1 \sum_{j=0}^{\ell} u_j F_{j,k} \int_0^a \rho L_m(r, \rho) \Phi_j^{(m)}(\rho) d\rho + \lambda_2 \int_0^a \rho L_m(r, \rho) \Phi_{\ell}^{(m)}(\rho) d\rho = f_{\ell}^{(m)}(r), \quad (4.15)$$

where

$$L_m(r, \rho) = \int_{-\pi}^{\pi} \frac{\cos m\psi d\psi}{\sqrt{r^2 + \rho^2 - 2r\rho \cos \theta}}. \quad (4.16)$$

Using the following relations, see [10, 14]:

$$\begin{aligned} (1) \quad & \int_0^{2\pi} \frac{\cos m\psi d\psi}{[1 - 2z \cos \psi + z^2]^{\alpha}} = \frac{2\pi(\alpha)_m z^m}{m!} F\left(\alpha, m + \alpha, m + 1, z^2\right), \\ & \left(|z| < 1, \operatorname{Re} \alpha > 0, (\alpha)_m = \frac{\Gamma(m + \alpha)}{\Gamma(\alpha)}\right); \\ (2) \quad & F\left(\alpha, \alpha + \frac{1}{2} - \beta, \beta + \frac{1}{2}, z^2\right) = (1 + z)^{-2\alpha} F\left(\alpha, \beta; 2\beta; \frac{4z}{1 + z^2}\right); \\ (3) \quad & \int_0^{\infty} J_{\alpha}(ax) J_{\alpha}(bx) x^{-\beta} dx = \frac{2^{-\beta} a^{\alpha} b^{\alpha} \Gamma\left(\alpha + \frac{1 - \beta}{2}\right)}{(a + b)^{2\alpha - \beta + 1} \Gamma(1 + \alpha) \Gamma\left(\frac{1 + \beta}{2}\right)} \\ & \times F\left(\alpha + \frac{1 - \beta}{2}, \alpha + \frac{1}{2}; 2\alpha + 1; \frac{4ab}{(a + b)^2}\right), \end{aligned}$$

where $F(a, b; c; z)$ is the Gauss hypergeometric function, $\Gamma(x)$ is the Gamma function, and $J_n(x)$ is the Bessel function, the kernel (4.16) takes the form

$$L_m(r, \rho) = 2\pi \int_0^{\infty} J_m(u\rho) J_m(ur) du. \quad (4.17)$$

Using (4.17) in (4.15), we obtain

$$\lambda_1 \sum_{j=0}^{\ell} u_j F_{j,k} \int_0^a K_m(r, \rho) Z_j^{(m)}(\rho) d\rho + \lambda_2 \int_0^a K_m(r, \rho) Z_{\ell}^{(m)}(\rho) d\rho = g_{\ell}^{(m)}(r),$$

$$\left(Z_j^{(m)}(r) = \sqrt{r} \Phi_j^{(m)}, \quad g_{\ell}^{(m)} = \sqrt{r} f_{\ell}^{(m)}(r) \right), \quad (4.18)$$

where

$$K_m(r, \rho) = 2\pi\sqrt{r\rho} \int_0^{\infty} J_m(u\rho) J_m(ur) du. \quad (4.19)$$

Equation (4.18) represents a system of Fredholm integral equations of the first kind with kernel (4.19) takes a form of Weber-Sonien integral formula.

Assume the solution of (4.18) at $\alpha = 1$, in the form

$$Z_k^{(m)}(r) = \frac{1}{\sqrt{1-r^2}} \sum_{n_k=0}^{\infty} a_{n_k}^{(m)} P_{2n_k}^{(m)}(\sqrt{1-r^2}), \quad (k = 0, 1, 2, \dots, \ell), \quad (4.20)$$

where $P_{2n}(y)$ is the Legendre polynomial. Then, using the same way of [5, 6], we obtain

$$\begin{aligned} & \lambda_1 \sum_{j=0}^{\ell} u_j F_{j,\ell} \int_0^1 \frac{K_m(r, \rho) P_{m_j}^{(n, -\frac{1}{2})}(1-2\rho^2) d\rho}{\sqrt{1-\rho^2}} \\ & + \lambda_2 \int_0^1 \frac{K_m(r, \rho) P_{m_{\ell}}^{(n, -\frac{1}{2})}(1-2\rho^2) d\rho}{\sqrt{1-\rho^2}} \\ & = \lambda_1 r^n \sum_{j=0}^{\ell} \mu_{m_j} u_j F_{j,\ell} P_{m_j}^{(n, -\frac{1}{2})}(1-2r^2) + \lambda_2 r^n \mu_{m_{\ell}} P_{m_{\ell}}^{(n, -\frac{1}{2})}, \end{aligned} \quad (4.21)$$

where, in general,

$$u_{m_k} = \frac{\Gamma^2\left(\frac{1}{2} + m_k\right)}{(2m_k)! \Gamma(1 + nm_k)}, \quad (4.22)$$

and $P_m^{(\alpha, \beta)}(x)$ is a Jacobi polynomial.

4.2. Generalized potential kernel in finite domain

When the modules of the elasticity of the contact problem is changing according to the power law $\sigma_i = K_0 \varepsilon_i^v$, $0 \leq v < 1$, where σ_i and ε_i are the stress and strain rate intensities, respectively, while K_0 and v are the physical constants, see [7].

For this, the kernel of Equation (3.7) takes the form

$$K(x - \xi, y - \eta) = \left[(x - \xi)^2 + (y - \eta)^2 \right]^{-v}, \quad 0 \leq v < 1. \quad (4.23)$$

The kernel of Equation (4.23) is called the *generalized potential kernel*.

Using (4.23) in (3.7) and following the same steps of potential kernel, where $\Omega = \{(x, y, z) \in \Omega : \sqrt{x^2 + y^2} \leq a, z = 0\}$, we can arrive to the following results:

$$\lambda_1 \sum_{j=0}^{\ell} u_j F_{j,k} \int_0^1 K_m^{(v)}(r, \rho) Z_j^{(m)}(\rho) d\rho + \lambda_2 \int_0^1 K_m^{(v)}(r, \rho) Z_\ell^{(m)}(\rho) d\rho = g_\ell^{(m)}(r), \quad (4.24)$$

where

$$K_m^{(v)}(r, \rho) = c \sqrt{r\rho} \int_0^\infty u^{2v-1} J_m(u\rho) J_m(ur) du, \quad (c = \frac{\pi \Gamma(1-v) \cdot 2^{2(1-v)}}{\Gamma(v)}). \quad (4.25)$$

The generalized potential kernel of (4.25) takes a generalized form of Weber-Sonien integral formula. Representing the unknown functions $Z_j^{(m)}$ and the known functions $g_\ell^{(m)}(r)$ in the Jacobi polynomials form

$$Z_k^{(m)}(r) = \frac{1}{(1-r^2)^\sigma} \sum_{n_k=0}^{\infty} a_{n_k}^{(m)} \cdot {}^{(m)}P_{n_k}^{(m, -\sigma)}(1-2r^2), \quad (4.26)$$

$$g_k^{(m)}(r) = \frac{1}{(1-r^2)^\sigma} \sum_{n_k=0}^{\infty} g_{n_k}^{(m)} \cdot {}^{(m)}P_{n_k}^{(m, -\sigma)}(1-2r^2), \quad (4.27)$$

then using Krein's method, see [7], we can obtain the following spectral relationships:

$$\begin{aligned} & \lambda_1 \sum_{j=0}^{\ell} \int_0^{\ell} \frac{u^{1+m} K_m^v(u, v)}{(1-u^2)^{\sigma^-}} \cdot {}^{(m)}P_{n_j}^{(m, -\sigma^-)}(1-2u^2) du \\ & + \lambda_2 \int_0^1 u^{1+m} K_m^{(v)}(u, v) \cdot {}^{(m)}P_{n_\ell}^{(m, -\sigma^-)}(1-2u^2) du \\ & = \lambda_1 \sum_{j=0}^{\ell} u_j \mu_{n_j} F_{j,k} \cdot {}^{(m)}P_{n_j}^{(m, -\sigma^-)}(1-2u^2) + \lambda_2 u_{n_\ell} \cdot {}^{(m)}P_{n_\ell}^{(m, -\sigma^-)}(1-2u^2), \end{aligned} \quad (4.28)$$

where

$$\begin{aligned} \mu_{n_\ell} &= 2^{2\sigma^+} \Gamma(n_\ell + \sigma^+) \Gamma(2n_\ell + \sigma^+) [n_\ell! \Gamma(1+2n_\ell)]^{-1}, \\ (\sigma^\pm &= \frac{1 \pm w}{2}, \quad v = w + \frac{1}{2}, \quad 0 \leq w < \frac{1}{2}). \end{aligned}$$

Taking in mind the basic relations of Bessel function, we can prove that, the generalized potential kernel (4.25) satisfies the following non-homogeneous wave equation:

$$\begin{aligned} & \left(\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial \rho^2} \right) K_m^v(r, \rho) = (h(r) - h(\rho)) K_m^v(r, \rho), \\ & h(r) = \left(m^2 - \frac{1}{4} \right) r^{-2}, \quad \left(m \neq \pm \frac{1}{2} \right). \end{aligned} \quad (4.29)$$

Also, many special cases can be derived from (4.25)

(i) Carleman kernel, $m = \pm \frac{1}{2}$:

$$K_{\pm \frac{1}{2}}^v(u, v) = |u - v|^{-v} = c\sqrt{uv} \int_0^\infty t^{2v-1} J_{\pm \frac{1}{2}}(tu) J_{\pm \frac{1}{2}}(tv) dt. \quad (4.30)$$

(ii) Logarithmic kernel, $v = \frac{1}{2}$, $m = \pm \frac{1}{2}$:

$$K_{\pm \frac{1}{2}}^{\frac{1}{2}}(u, v) = 2\pi\sqrt{uv} \int_0^\infty J_{\pm \frac{1}{2}}(tu) J_{\pm \frac{1}{2}}(tv) dt = -\ln|u - v|. \quad (4.31)$$

(iii) Elliptic kernel, $v = \frac{1}{2}$, $m = 0$:

$$K_0^{\frac{1}{2}}(u, v) = \frac{2\sqrt{uv}}{\pi(u+v)} E \frac{\sqrt{2uv}}{u+v} = 2\pi \int_0^\infty J_0(tu) J_0(tv) dt. \quad (4.32)$$

(iv) Potential kernel, $v = \frac{1}{2}$, see Equation (4.19).

The spectral relationships for the complete elliptic kernel can be obtained in the form

$$\begin{aligned} & \lambda_1 \sum_{j=0}^{\ell} u_j F_{j, \ell} \int_0^\ell \frac{u E\left(\frac{2\sqrt{r\rho}}{r+\rho}\right)}{\sqrt{1-\rho^2}} P_{2m_j}\left(\sqrt{1-\rho^2}\right) d\rho \\ & \quad + \lambda_2 \int_0^1 \frac{u E\left(\frac{2\sqrt{r\rho}}{r+\rho}\right)}{\sqrt{1-\rho^2}} P_{2m_\ell}\left(\sqrt{1-\rho^2}\right) d\rho \\ & = \lambda_1 \frac{\pi^2}{4} \sum_{j=0}^{\infty} u_j F_{j, \ell} \left[\frac{(2m_j - 1)!!}{(2m_j)!!} \right] P_{2m_j}\left(\sqrt{1-r^2}\right) \\ & \quad + \lambda_2 \frac{\pi^2}{4} \left[\frac{(2m_\ell - 1)!!}{(2m_\ell)!!} \right] P_{2m_\ell}\left(\sqrt{1-r^2}\right), \\ & (P_m^{(0, -\frac{1}{2})}(1 - 2x^2) = P_{2m}\left(\sqrt{1-x^2}\right), P_m(z) \text{ is a Legendre polynomial}). \end{aligned} \quad (4.33)$$

The importance of the integral equation with complete elliptic kernel came from the work of Kovalenko [11], who developed the Fredholm integral equation of the first kind for the mechanics mixed problem of continuous media, and obtained an approximate solution for the Fredholm integral equation of the first kind with complete elliptic kernel.

Many important spectral relations can be derived and established from the formula (4.27), for different values of v , $0 \leq v < 1$ and for higher order m_j , $j = 0, 1, 2, \dots, \ell$.

4.3. Potential kernel in a half-space

Assume in Equation (3.7), $k(x - \xi, y - \eta) = \left[(x - \xi)^2 + (y - \eta)^2 \right]^{-\frac{1}{2}}$,

$$\Omega = \{ (x, y, z) \in \Omega : -\infty < x < \infty, |y| \leq 1, -\infty < z \leq 0 \},$$

hence, we have the following system of integral equations:

$$\lambda_1 \sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-1}^1 K_0(\alpha|u-v|) \Phi_j^{(\alpha)}(v) dv + \int_{-1}^1 K_0(\alpha|u-v|) \Phi_{\ell}^{(\alpha)}(v) dv = f_{\ell}^{(\alpha)}(u). \quad (4.34)$$

In Equation (4.34), we used the following Fourier integral transformation, see [9]:

$$\begin{aligned} \Phi_j^{(\alpha)}(y) &= \int_{-\infty}^{\infty} \Phi_j(x, y) e^{i\alpha x} dx, \\ f_{\ell}^{(\alpha)}(y) &= \int_{-\infty}^{\infty} f_{\ell}(x, y) e^{i\alpha x} dx, \quad (i = \sqrt{-1}), \end{aligned} \quad (4.35)$$

and the definition of Macdonald kernel, see [5]

$$K_0(|\alpha| |x - \xi|) = \int_0^{\infty} \frac{\cos \alpha y dy}{\sqrt{(x - \xi)^2 + y^2}}, \quad (4.36)$$

where α is the Fourier parameter and K_0 is the Macdonald kernel.

Assuming the unknown functions $\check{\Phi}_J(u)$ in the Mathieu functions, see [9], we have the following spectral relationships:

$$\begin{aligned}
& \lambda_1 \sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-1}^1 \frac{C_{en_j}\left(\cos^{-1} u, -\frac{1}{4}\right)}{\sqrt{1-u^2}} K_0(\alpha|u-v|) du \\
& \quad + \lambda_2 \int_{-1}^1 \frac{C_{en_\ell}\left(\cos^{-1} u, -\frac{1}{4}\right)}{\sqrt{1-u^2}} K_0(\alpha|u-v|) du \\
& = \pi \lambda_1 \sum_{j=0}^{\ell} \frac{u_j F_{j,\ell} F_e K_{n_j}\left(0, -\frac{1}{4}\right) C_{en_j}\left(\cos^{-1} v, -\frac{1}{4}\right)}{F_e K_{n_j}^1\left(0, \frac{1}{4}\right)} \\
& \quad + \pi \lambda_2 \frac{F_e K_{n_\ell}\left(0, -\frac{1}{4}\right) C_{en_\ell}\left(\cos^{-1} v, -\frac{1}{4}\right)}{F_e K_{n_\ell}^1\left(0, \frac{1}{4}\right)}, \quad (4.37)
\end{aligned}$$

where $F_e K_{n_j}(r, -q)$ and $C_{en_\ell}(\theta, -q)$ are called the *Mathieu functions*

under the conditions $0 \leq \theta \leq 2\pi$, $\gamma < \infty$, $q = \frac{a^2}{4}$, and $a = 1$.

If the domain of integration in Equation (3.7) is considered as $\Omega = \{(x, y, z) \in \Omega : -\infty < x, y < \infty; z \leq 0\}$, and the kernel takes the potential function form, we will have the following:

$$\begin{aligned}
& \lambda_1 \sum_{j=0}^{\ell} u_j F_{j,\ell} \int_0^\infty \frac{K_0(|u-v|)}{\sqrt{v}} e^{-v} L_{m_j}^{-\frac{1}{2}}(2v) dv \\
& \quad + \lambda_2 \int_0^\infty \frac{K_0(|u-v|)}{\sqrt{v}} e^{-v} L_{m_\ell}^{-\frac{1}{2}}(2v) dv \\
& = \frac{\lambda_1 \pi}{\sqrt{2}} \sum_{j=0}^{\ell} u_j F_{j,\ell} \frac{[(2m_j - 1)!]}{(2m_j)!} e^{-u} L_{m_j}^{-\frac{1}{2}}(2u)
\end{aligned}$$

$$+ \frac{\lambda_2 \pi}{\sqrt{2}} \frac{[(2m_\ell - 1)!]}{(2m_\ell)!} e^{-u} L_{m_\ell}^{-\frac{1}{2}}(2u), \quad (u \geq 0), \quad (4.38)$$

where $L_m^\sigma(x)$ is the Chebyshev-Laguerre polynomial.

4.4. Generalized Macdonald kernel

If we take in (3.7)

$$K(x - \xi, y - \eta) = \left[(x - \xi)^2 + (y - \eta)^2 \right]^{-(\mu + \frac{1}{2})}, \quad 0 \leq \mu < \frac{1}{2},$$

$\Omega = \{ (x, y, z) \in \Omega : -\infty < x, y < \infty; -\infty < z \leq 0 \}$, then using the Fourier transformation of Equation (4.25), we have the following system:

$$\begin{aligned} \lambda_1 \sum_{j=0}^{\ell} u_j F_{j,\ell} \int_{-\infty}^{\infty} \frac{K_\mu | \alpha ||u - v| \Phi_j^{(\alpha)}(v) dv}{|u - v|^\mu} \\ + \lambda_2 \int_{-\infty}^{\infty} \frac{K_\mu | \alpha ||u - v| \Phi_\ell^{(\alpha)}(v) dv}{|u - v|^\mu} = g_\ell(u), \end{aligned} \quad (4.39)$$

$$g_\ell(u) = \pi^{-\frac{1}{2}} 2^{\mu-1} \Gamma\left(\frac{1}{2} + \mu\right) (\alpha)^{-\mu} f_\ell^{(\alpha)}(u),$$

where $K_n(| \alpha || \cdot |)$, $\alpha > 0$, is the generalized Macdonald kernel, that has the following expansion:

$$\frac{K_\mu(|u - v|)}{|u - v|^\mu} = \frac{2^r \sqrt{\pi}}{e^{u+v}} \sum_{n=0}^{\infty} L_n^{\mu-\frac{1}{2}}(2u) L_n^{\mu-\frac{1}{2}}(2v), \quad \alpha = 1, \quad (4.40)$$

where $L_n^\sigma(2x)$ is the Chebyshev-Laguerre polynomial.

Using (4.40) in (4.39), then expanding the unknown functions $\Phi_j^{(\alpha)}(u)$ and the known functions $g_\ell(u)$ in terms of Chebyshev-Laguerre polynomials, we obtain

$$\lambda_1 \sum_{j=0}^{\ell} u_j F_{j,\ell} \int_0^\infty \frac{K_\mu(|u - v|)}{|u - v|^\mu v^{\frac{1}{2}-\mu}} L_{m_j}^{\mu-\frac{1}{2}}(2v) dv + \lambda_2 \int_0^\infty \frac{K_\mu(|u - v|)}{|u - v|^\mu v^{\frac{1}{2}-\mu}} L_{m_\ell}^{\mu-\frac{1}{2}}(2v) dv$$

$$\begin{aligned}
&= \lambda_1 \sqrt{\frac{\pi}{2}} e^{-u} \sum_{j=0}^{\ell} \frac{u_j F_{j,\ell}}{(m_j)!} \Gamma\left(\frac{1}{2} - \mu\right) \Gamma\left(\frac{1}{2} + \mu + m_j\right) L_{m_j}^{\mu-\frac{1}{2}}(2u) \\
&\quad + \lambda_2 \frac{\sqrt{\frac{\pi}{2}} e^{-u}}{(m_j)!} \Gamma\left(\frac{1}{2} - \mu\right) \Gamma\left(\frac{1}{2} + \mu + m_\ell\right) L_{m_j}^{\mu-\frac{1}{2}}(2u), \\
&\quad (u \geq 0, m_j \geq 0, \quad j = 0, 1, \dots, \ell), \tag{4.41}
\end{aligned}$$

where $\Gamma(\cdot)$ is the Gamma function.

If we let, in Equation (4.41), $\mu = 0$, directly we have the result of the formula (4.38).

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